## Proof that is analytic

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The product and composition of analytic functions are analytic. Using this, it can be shown is analytic for any constant complex number using the Cauchy-Riemann equations. If can be shown to be analytic for any real , and for any imaginary , then it can be shown for any complex since .

Begin by solving for real :

Now, additional conversions to exponentials are helpful

To get a longer form of

Noticing that ,

Moving on to ,

Therefore, the first of the Cauchy-Riemann equations has been solved:

Moving to the next equation, the above patterns can be reused to move more quickly:

Now, both Cauchy-Riemann equations are shown to be true and to be analytic for real (except at zero since has an essential singularity there). The only thing that remains is to show the same result for imaginary . Notice that this can be shown by taking , where is a real number. We already know is analytic, and if  is analytic, is the composition of analytic functions, and therefore analytic. This is proven by showing is analytic: